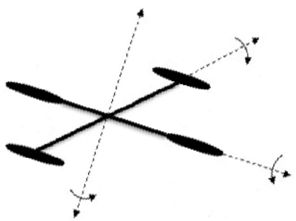
7장 Vectors

7.3 내적



- 내적의 정의

정의 7.3 **두 벡터의 내적**

두 벡터 **a**와 **b**의 내적은 스칼라

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \qquad (1)$$

이다. 여기서 θ는 두 벡터 사이의 각으로 그 범위는 $0 \le \theta \le \pi$이다.

(i) $\mathbf{a}\cdot\mathbf{b}=0$ if $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$

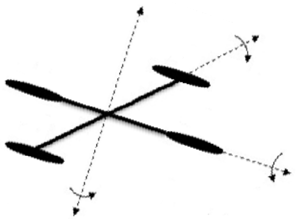
(ii) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ (교환법칙)

(iii) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ (분배법칙)

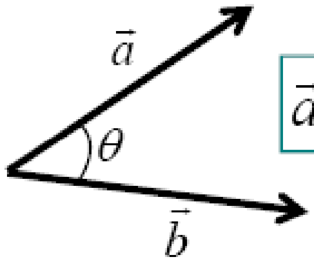
(iv) $\mathbf{a}\cdot(k\mathbf{b})=(k\mathbf{a})\cdot\mathbf{b}=k(\mathbf{a}\cdot\mathbf{b})$ k는 스칼라

(v) $\mathbf{a}\cdot\mathbf{a}\ge 0$

(vi) $\mathbf{a}\cdot\mathbf{a}=\|\mathbf{a}\|^2$



- 내적의 정의



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a} = [a_1, a_2, a_3],\ \vec{b} = [b_1, b_2, b_3]$$

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k},\quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$\vec{a}\cdot\vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right)\cdot\left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right)$$

$$= a_1b_1\vec{i}\cdot\vec{i} + a_1b_2\vec{i}\cdot\vec{j} + a_1b_3\vec{i}\cdot\vec{k}$$

$$+ a_2b_1\vec{j}\cdot\vec{i} + a_2b_2\vec{j}\cdot\vec{j} + a_2b_3\vec{j}\cdot\vec{k}$$

$$+ a_3b_1\vec{k}\cdot\vec{i} + a_3b_2\vec{k}\cdot\vec{j} + a_3b_3\vec{k}\cdot\vec{k}$$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

자신과의 내적은 $\theta = 0^\circ$

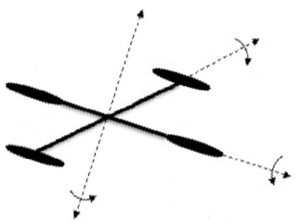
$$\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos(0^\circ) = |\vec{a}|^2$$

두 벡터가 직교한다면 $\theta = 90^\circ$

$$\vec{a}\cdot\vec{b} = 0$$

$$\vec{i}\cdot\vec{i},\ \vec{j}\cdot\vec{j},\ \vec{k}\cdot\vec{k} = 1$$

$$\vec{i}\cdot\vec{j},\ \vec{i}\cdot\vec{k},\ \vec{j}\cdot\vec{k} = 0$$



예제 2 (4)를 이용한 내적

$\mathbf{a}=10\mathbf{i}+2\mathbf{j}-6\mathbf{k}$, $\mathbf{b}=-1/2\mathbf{i}+4\mathbf{j}-3\mathbf{k}$ 라 하면, (4)로부터

$$\mathbf{a}\cdot\mathbf{b}=(10)\left(-\frac{1}{2}\right)+(2)(4)+(-6)(-3)=21$$

이다. ❑

- 직교의 판정

정리 7.1 **직교벡터의 판정**

영벡터가 아닌 두 벡터 **a** 와 **b** 가 직교하기 위한 필요충분조건은 $\mathbf{a} \cdot \mathbf{b} = 0$ 이다.

예제 4 **직교벡터**

$\mathbf{a} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 14\mathbf{j} + 5\mathbf{k}$ 라 하면

$$\mathbf{a} \cdot \mathbf{b} = (-3)(2) + (-1)(14) + (4)(5) = 0$$

이므로 정리 7.1 에 의하여 **a** 와 **b** 는 직교한다. ❑

- 두 벡터 사이의 각

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

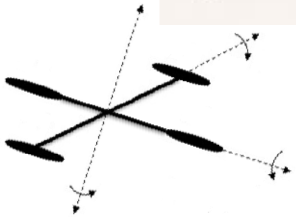
예제 5 두 벡터 사이의 각

$\mathbf{a}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ 와 $\mathbf{b}=-1\mathbf{i}+5\mathbf{j}+\mathbf{k}$ 사이의 각을 구하라.

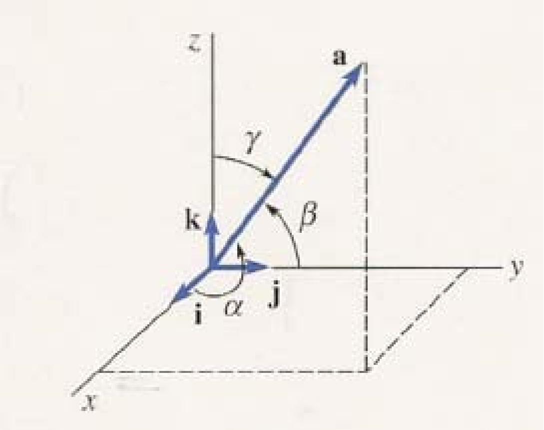
풀이 $\|\mathbf{a}\|=\sqrt{14}$, $\|\mathbf{b}\|=\sqrt{27}$, $\mathbf{a}\cdot\mathbf{b}=14$ 와 (6)으로부터

$$\cos\theta = \frac{14}{\sqrt{14}\sqrt{27}} = \frac{\sqrt{42}}{9}$$

이므로 $\theta = \cos^{-1}\left(\frac{\sqrt{42}}{9}\right) \approx 0.77$ radian 또는 $\theta \approx 44.9°$이다. ❑



- 방향 코사인



$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\|\mathbf{i}\|},\quad \cos\beta = \frac{\mathbf{a}\cdot\mathbf{j}}{\|\mathbf{a}\|\|\mathbf{j}\|},\quad \cos\gamma = \frac{\mathbf{a}\cdot\mathbf{k}}{\|\mathbf{a}\|\|\mathbf{k}\|}$$

$$\cos\alpha = \frac{a_1}{\|\mathbf{a}\|},\quad \cos\beta = \frac{a_2}{\|\mathbf{a}\|},\quad \cos\gamma = \frac{a_3}{\|\mathbf{a}\|}$$

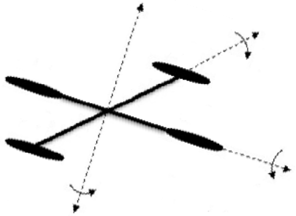
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

예제 6 방향 코사인/각

벡터 $\mathbf{a}=2\mathbf{i}+5\mathbf{j}+4\mathbf{k}$의 방향 코사인과 방향각을 구하라.

풀이 $\|\mathbf{a}\|=\sqrt{2^2+5^2+4^2}=\sqrt{45}=3\sqrt{5}$로부터, 방향 코사인은

$$\cos\alpha = \frac{2}{3\sqrt{5}},\quad \cos\beta = \frac{5}{3\sqrt{5}},\quad \cos\gamma = \frac{4}{3\sqrt{5}}$$

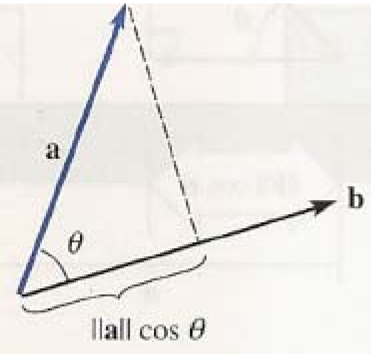


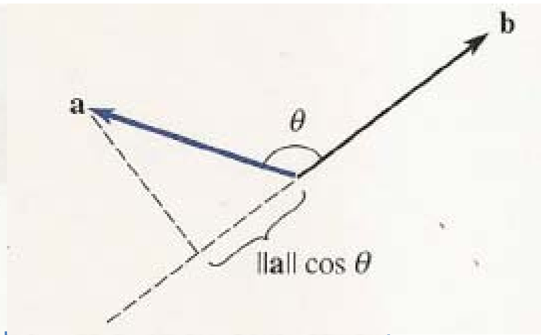
- b에서의 a의 성분

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$$

$$a_1 = \mathbf{a}\cdot\mathbf{i}, \quad a_2 = \mathbf{a}\cdot\mathbf{j}, \quad a_3 = \mathbf{a}\cdot\mathbf{k}$$

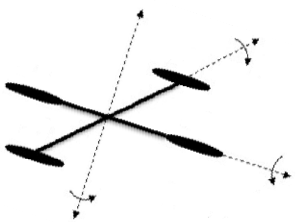
$$\text{comp}_{\mathbf{i}}\mathbf{a} = \mathbf{a}\cdot\mathbf{i}, \quad \text{comp}_{\mathbf{j}}\mathbf{a} = \mathbf{a}\cdot\mathbf{j}, \quad \text{comp}_{\mathbf{k}}\mathbf{a} = \mathbf{a}\cdot\mathbf{k}$$





$$\text{comp}_{\mathbf{b}}\mathbf{a} = \|\mathbf{a}\|\cos\theta$$

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\left(\frac{1}{\|\mathbf{b}\|}\mathbf{b}\right) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}$$



예제 7 한 벡터에서의 다른 벡터의 성분

$\mathbf{a}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $\mathbf{b}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ 일 때, $\text{comp}_{\mathbf{b}}\mathbf{a}$ 와 $\text{comp}_{\mathbf{a}}\mathbf{b}$ 를 구하라.

풀이 $\mathbf{b}$ 방향의 단위벡터는

$$\|\mathbf{b}\| = \sqrt{6}, \quad \frac{1}{\|\mathbf{b}\|}\mathbf{b} = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

이다. 따라서 (10)으로부터

$$\text{comp}_{\mathbf{b}}\mathbf{a} = (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -\frac{3}{\sqrt{6}}$$

이다. 따라서 (10)을 이용하면

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{b} \cdot \left(\frac{1}{\|\mathbf{a}\|}\mathbf{a}\right)$$

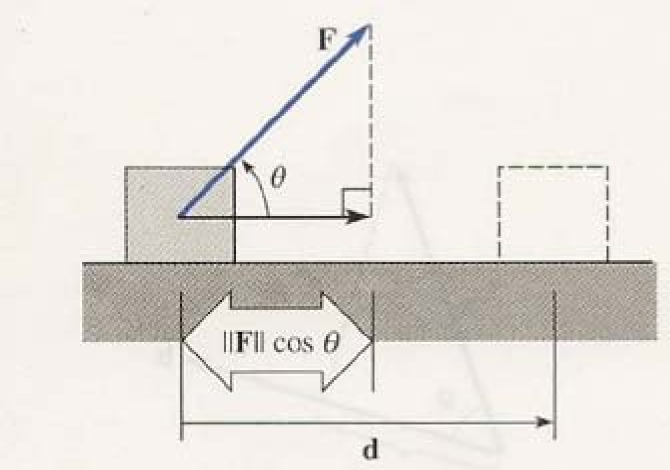
$$\|\mathbf{a}\| = \sqrt{29}, \quad \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

이고

$$\text{comp}_{\mathbf{a}}\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -\frac{3}{\sqrt{29}}$$

이다. ❑

- 내적의 물리적 해석



$$W = (\|\mathbf{F}\| \cos\theta)\, \|\mathbf{d}\| = \|\mathbf{F}\|\, \|\mathbf{d}\| \cos\theta$$

$$W = \mathbf{F} \cdot \mathbf{d}$$

예제 8 일정한 힘이 한 일

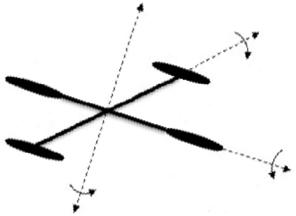
일정한 힘 $\mathbf{F}=2\mathbf{i}+4\mathbf{j}$가 블록에 작용하여 블록이 점 $P_1(1, 1)$에서 $P_2(4, 6)$으로 이동할 때, $\mathbf{F}$가 한 일을 구하라. $\|\mathbf{F}\|$와 $\|\mathbf{d}\|$의 단위는 각각 N(뉴턴)과 m(미터)이다.

풀이 블록의 변위는

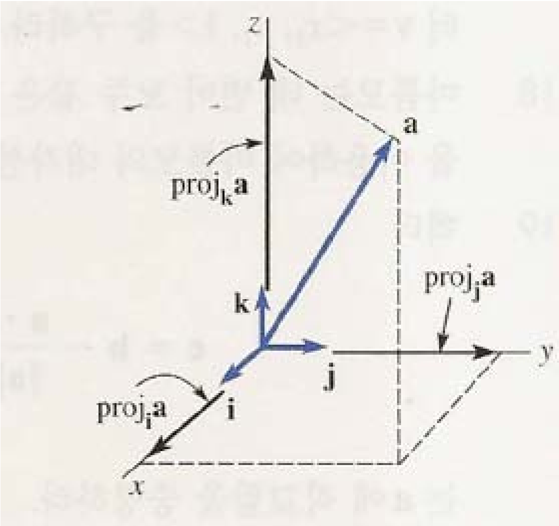
$$\mathbf{d} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = 3\mathbf{i} + 5\mathbf{j}$$

로 주어진다. (11)로부터 한 일은

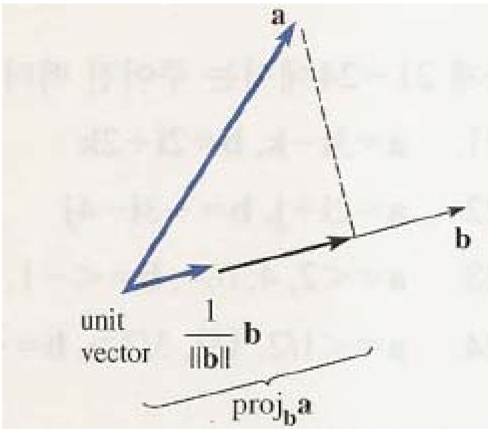
$$W = (2\mathbf{i} + 4\mathbf{j}) \cdot (3\mathbf{i} + 5\mathbf{j}) = 26 \text{ N-m}$$



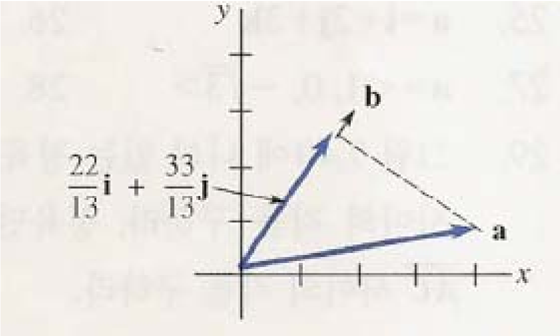
- b 위로의 a의 사영



$$\text{proj}_{\mathbf{i}}\mathbf{a} = (\text{comp}_{\mathbf{i}}\mathbf{a})\mathbf{i} = (\mathbf{a}\cdot\mathbf{i})\mathbf{i} = a_1\mathbf{i}$$



$$\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a})\left(\frac{1}{\|\mathbf{b}\|}\mathbf{b}\right) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$$



예제 9 한 벡터의 다른 벡터 위로의 사영

벡터 $\mathbf{b}=2\mathbf{i}+3\mathbf{j}$ 위로의 $\mathbf{a}=4\mathbf{i}+\mathbf{j}$의 사영을 구하고 그래프를 그리라.

풀이 먼저 $\mathbf{a}$와 $\mathbf{b}$의 성분을 구한다. $\|\mathbf{b}\|=\sqrt{13}$이므로, (10)으로부터

$$\text{comp}_{\mathbf{b}}\mathbf{a} = (4\mathbf{i}+\mathbf{j})\cdot\frac{1}{\sqrt{13}}(2\mathbf{i}+3\mathbf{j}) = \frac{11}{\sqrt{13}}$$

이며, (12)로부터

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{11}{\sqrt{13}}\right)\left(\frac{1}{\sqrt{13}}\right)(2\mathbf{i}+3\mathbf{j}) = \frac{22}{13}\mathbf{i} + \frac{33}{13}\mathbf{j}$$

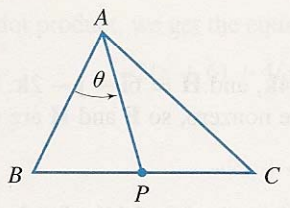


FIGURE 5.20

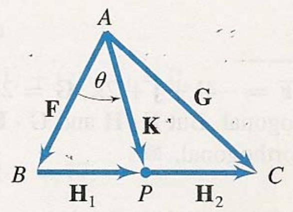


FIGURE 5.21

EXAMPLE 5.7

The points $A\ (1, -2, 1)$, $B\ (0, 1, 6)$, and $C\ (-3, 4, -2)$ form the vertices of a triangle. Suppose we want the angle between the line $\overline{AB}$ and the line from A to the midpoint of $\overline{BC}$. This line is a median of the triangle and is shown in Figure 5.20

Visualize the sides of the triangle as vectors, as in Figure 5.21. If P is the midpoint of $\overline{BC}$, then $\mathbf{H}_1 = \mathbf{H}_2$ because both vectors have the same direction and length. From the coordinates of the vertices, calculate

$$\mathbf{F} = -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \quad \text{and} \quad \mathbf{G} = -4\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}.$$

We want the angle between $\mathbf{F}$ and $\mathbf{K}$, so we need $\mathbf{K}$. By the parallelogram law,

$$\mathbf{F} + \mathbf{H}_1 = \mathbf{K} \quad \text{and} \quad \mathbf{K} + \mathbf{H}_2 = \mathbf{G}.$$

Since $\mathbf{H}_1 = \mathbf{H}_2$, these equations imply that

$$\mathbf{K} = \mathbf{F} + \mathbf{H}_1 = \mathbf{F} + (\mathbf{G} - \mathbf{K}).$$

Therefore,

$$\mathbf{K} = \frac{1}{2}(\mathbf{F} + \mathbf{G}) = -\frac{5}{2}\mathbf{i} + \frac{9}{2}\mathbf{j} + \mathbf{k}.$$

Now the cosine of the angle we want is

$$\cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{K}}{\|\mathbf{F}\|\|\mathbf{K}\|} = \frac{42}{\sqrt{35}\sqrt{110}} = \frac{42}{\sqrt{3850}}.$$

θ is approximately 0.83 radians. ■

DEFINITION 5.6 ***Orthogonal Vectors***

Vectors **F** and **G** are orthogonal if and only if $\mathbf{F} \cdot \mathbf{G} = 0$.

EXAMPLE 5.8

Let $\mathbf{F} = -4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{G} = 2\mathbf{i} + 4\mathbf{k}$, and $\mathbf{H} = 6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Then $\mathbf{F} \cdot \mathbf{G} = 0$, so **F** and **G** are orthogonal. But $\mathbf{F} \cdot \mathbf{H}$ and $\mathbf{G} \cdot \mathbf{H}$ are nonzero, so **F** and **H** are not orthogonal, and **G** and **H** are not orthogonal. ■

EXAMPLE 5.9

Two lines are given parametrically by

$$L_1: x = 2 - 4t, \quad y = 6 + t, \quad z = 3t$$

and

$$L_2: x = -2 + p, \quad y = 7 + 2p, \quad z = 3 - 4p.$$

We want to know whether these lines are perpendicular. (It does not matter whether the lines intersect).

The idea is to form a vector along each line and test these vectors for orthogonality. For a vector along L_1, choose two points on this line, say $(2, 6, 0)$ when $t = 0$ and $(-2, 7, 3)$ when $t = 1$. Then $\mathbf{F} = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ is along L_1. Two points on L_2 are $(-2, 7, 3)$ when $p = 0$ and $(-1, 9, -1)$ when $p = 1$. Then $\mathbf{G} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ is along L_2. Since $\mathbf{F} \cdot \mathbf{G} = -14 \neq 0$, these vectors, hence these lines, are not orthogonal. ■

EXAMPLE 5.10

Suppose we want the equation of a plane Π containing the point $(-6, 1, 1)$ and perpendicular to the vector $\mathbf{N} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

A strategy to find such an equation is suggested by Figure 5.22. A point (x, y, z) is on Π if and only if the vector from $(-6, 1, 1)$ to (x, y, z) is in Π and therefore is orthogonal to $\mathbf{N}$. This means that

$$((x+6)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k}) \cdot \mathbf{N} = 0.$$

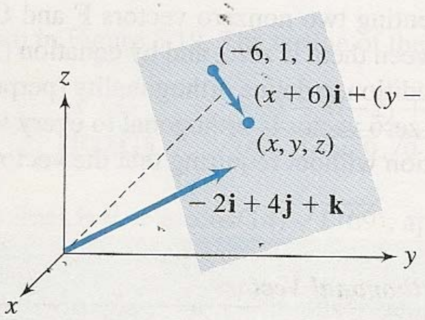


FIGURE 5.22

Carrying out this dot product, we get the equation

$$-2(x+6) + 4(y-1) + (z-1) = 0,$$

or

$$-2x + 4y + z = 17.$$

This is the equation of the plane. Of course the given point $(-6, 1, 1)$ satisfies this equation. ■